NUMERICAL METHODS FOR SOLVING THE NON-LINEAR PROBLEM OF A WIND-DRIVEN, HOMOGENEOUS, EQUATORIAL UNDERCURRENT

by

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ABSTRACT

The non-linear problem of a wind-driven, homogeneous, equatorial undercurrent is solved numerically for a small value of the kinematic eddy viscosity coefficient. Various finite difference schemes are discussed. A boundary-layer type profile is obtained for this small viscosity, and a study is made of the balance of forces in the different regions of flow.

Thesis Supervisor: Jule G. Charney
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I. INTRODUCTION

The Cromwell current is an equatorial phenomenon observed in the Pacific Ocean. Similar currents are observed in the Atlantic and , depending on season, in the Indian Ocean. In the equatorial region of the Pacific there is an east wind blowing over the ocean surface and a thin, swift flowing countercurrent beneath the surface. It is remarkably stable and extends over a considerable distance of about 5000km. west of the Galapagos Islands. To the east of these islands the current breaks up, probably owing to the reversal of the west-east pressure gradient force.

On the 1961 Swan Song expedition, (Knauss 1966) the core of the current was observed at a depth of between 50 and 100m.,its maximum speed was 100 to 150 $cm.sec.^{-1}$ with a surface westward velocity of 35 to 120 $cm.sec.^{-1}$ The vertical velocity field cannot be observed directly, but there are indications from salinity, temperature, oxygen, phosphate and carbon dioxide measurements of upwelling near the surface and descending motion below the countercurrent core. The magnitude of the vertical velocity estimated from considerations of the meridional circulation was 0.5 to $5*10^{-3} cm.sec.^{-1}$ The current is confined within about 2° either side of the equator, and is fairly symmetrical about the equator. A westward pressure gradient force of order 10^{-5} dyne $cm.^{-1}$ was observed. This force probably serves to maintain the angular momentum of the current against the frictional torque exerted by the surrounding slower moving waters.

At the equator there is no vertical component of the earth's rotation vector, and therefore a normal geostrophic balance cannot apply here. Abalance then must be obtained between the pressure gradient, inertial and viscous forces, Following Charney (1960) certain approximations, suggested by the above empircal facts, are applied to the governing equations of motion, and the resulting parabolic system is solved numerically using finite differences. The accuracy and rate of convergence of various finite difference schemes is compared.

The model contains a free eddy viscosity parameter, assumed constant. Unfortunately the system is very sensitive to the choice of this parameter. The kinematic eddy viscosity coefficient ν is not a constant in the vertical and is not reliably known. Assuming a parabolic form for u, Knauss estimates ν to be about 5cm.sec.$^{-2}$ Charney (1960) determined u and w velocity profiles for various values of this parameter and obtained qualitative agreement with the observed u-field for a value of ν of about 17cm.sec.$^{-2}$ However with this model it is impossible to obtain, using any one value for ν , both a westward surface current and an eastward current beneath the surface of sufficiently large magnitude to be comparable with observation. The agreement is only qualitative.

Here this ideal system is solved numerically for a reduced viscosity coefficient. The u-velocity profile demonstrates certain boundary-layer characteristics, and the balance of forces in the different regions of the flow is investigated in an attempt to form an approximate analytic solution.

II. THE GOVERNING EQUATIONS AND THE ASSUMPTIONS ADOPTED IN THE MODEL

A steady, uniform wind stress is applied in the east-west direction to a homogeneous β -plane ocean of infinite horizontal extent. The flow is assumed symmetric about the equator and independent of longitude. Boundary conditions are applied, not at the bottom of the ocean, but at the average depth of the equatorial thermocline. This is necessary for the validity of the homogeneity assumption, and seems plausible because transmission of momentum across the thermocline is inhibited by its high gravitational stability, the thermocline therefore simulating a rigid boundary. Making use of the β -plane approximation, the equations of motion are

$$\rho \frac{du}{dt} - \rho\beta y v = -p_x + (\mu u_z)_z = -p_x + \tau_z^{(x)} \quad (1)$$

$$\rho \frac{dv}{dt} + \rho\beta y u = -p_y + (\mu v_z)_z = -p_y + \tau_z^{(y)} \quad (2)$$

$$p_z = 0$$

Here p is the departure of the pressure from the hydostatic value.

In the extra-equatorial flow the inertial terms are small and the equations reduce to

$$-\rho\beta y v = -p_x + \tau_z^{(x)} \qquad (3)$$

$$\rho\beta y u = -p_y + \tau_z^{(y)} \qquad (4)$$

together with the continuity equation

$$u_x + v_y + w_z = 0$$

On the assumptions of constant surface stress, negligible bottom stress and vertical velocity zero and negligible at the bottom and surface respectively, Charney (1960) has shown that the vertically integrated meridional transport over the total depth is zero. Applying this to a vertically integrated equation (3) gives $-H\,p_x = -(\tau^{(x)})_{z=H} = \tau$; τ is a positive number. This relation is derived for the extraequatorial region, but from arguments based on the uniformity of the current with longitude, Charney extends the region of application to the equator.

The steady-state equations governing the motion in the equatorial region may be written

$$v u_y + w u_z - \beta y v = P + \nu u_{zz} \qquad (5)$$

$$v v_y + w v_z + \beta y u = Q + \nu v_{zz} \qquad (6)$$

$$v_y + w_z = 0 \qquad (7)$$

where P and Q are the pressure gradient forces in the x and y directions respectively, and $P = \frac{\tau}{\rho H}$. Derivatives of the velocity fields with respect to x are assumed zero or very small from the observed uniformity with longitude. Due to symmetry about the equator, $v = 0$. Differentiating equation (6) with respect to y, substituting from the continuity equation (7) and putting $y = v = v_z = 0$, one obtains

$$w_z^2 - w w_{zz} + \beta u = Q_y - \nu w_{zzz} \qquad (8)$$

and from (5)

$$w u_z = P + \nu u_{zz} \qquad (9)$$

The system is then determined by equations (8) and (9) subject to the boundary conditions

$$\left.\begin{aligned} & u(0) = w(0) = v_y(0) = w_z(0) = 0 \\ & u_z(H) = \frac{\tau^{(x)}}{\rho\nu} = -\frac{\tau}{\rho H} = -\frac{PH}{\nu} \\ & v_z(H) = \frac{\tau^{(y)}}{\rho\nu} = 0 \Rightarrow w_{zz}(H) = 0 \\ & w(H) = 0 \end{aligned}\right\} \qquad (10)$$

The steady-state equations are very difficult to solve but it is comparatively easy to obtain steady-state solutions to the time-dependent problem using numerical methods. Time dependence is thus introduced into the system and equations (8) and (9) become

$$u_t + w u_z = P + \nu u_{zz} \quad (11)$$

$$w_{tz} - w_z^2 + w w_{zz} - \beta u = -Q_y + \nu w_{zzz} \quad (12)$$

For convenience equation (12) is integrated with respect to z giving

$$w_t - w w_z = G(z) - \frac{z}{H} G(H) + \nu w_{zz} \quad (13)$$

where $G(z) = \int_0^z (\beta u - 2 w w_{zz})\, dz - \nu w_{zz}(0)$ (14)

$$G(H) = H (Q_y)_{y=0}$$

The boundary conditions (10) have been utilized here.

For ease of comparison the equations were non-dimensionalized using the scaling factors of Charney, 1960. This scaling was found to be inappropriate for the case of a small viscosity coefficient and other scaling possibilities are discussed below. In equation (5) the assumption of dominance of the inertial over the viscous term leads to the choice of the following length and velocity scales.

$$V_0 = \left(\frac{P^2}{\beta}\right)^{1/3}, \quad W_0 = H(P\beta)^{1/3}, \quad L = \left(\frac{P}{\beta^2}\right)^{1/3}$$

The total depth H is used as the height scale. Non-dimensional quantities U, V, W, η and ζ are defined by

$$u = V_0 U, \quad v = V_0 V, \quad w = W_0 W, \quad y = L\eta, \quad z = H\zeta$$

Equations (11) and (12) then become

$$U_T + W U_\zeta = 1 + c\, U_{\zeta\zeta} \qquad (15)$$

$$W_{T\zeta} - W_\zeta^2 + W W_{\zeta\zeta} - U = Q'_\eta + c\, W_{\zeta\zeta\zeta}$$

where $c = \dfrac{\nu}{H^2 (P\beta)^{1/3}}$, the non-dimensional time is given by

$$T = t\,(P\beta)^{1/3}$$

and

$$Q'_\eta = Q_\eta P^{-1} = Q_y\,(P\beta)^{-2/3}$$

$$G(\zeta) = \frac{G(z)}{H(P\beta)^{2/3}}$$

$$G(1) = (Q'_\eta)_{\eta=0}$$

Equations (13) and (14) become

$$W_T - W W_\zeta = G(\zeta) - \zeta G(1) + c W_{\zeta\zeta} \qquad (16)$$

and

$$G(\zeta) = \int_0^\zeta (U - 2 W W_{\zeta\zeta})\, d\zeta - c\, W_{\zeta\zeta}^{(0)} \qquad (17)$$

The surface stress condition is given by

$$U_\zeta = -\frac{1}{c}$$

Finite difference approximations to equations (15) , (16) and (17) are formed as described later and the problem is treated as an initial value problem. The system is solved iteratively using the I.B.M. 7094 computer. Taking zero as the initial choice of u and w-fields, the velocity fields are allowed to evolve in time until, due to frictional dissipation in the system, they approach a steady-state sufficiently closely.

III. FINITE DIFFERENCE SCHEMES

In this section the variables are still non-dimensional but, for convenience, the dimensional notation is used.

A uniform grid is formed by dividing the depth into J equal intervals. The height of grid-point j is $j\,\Delta z$ where $\Delta z = \frac{1}{J}$ and j runs from 0 to J. Discrete times are defined by $t = n\,\Delta t,\ (n = 0, 1, 2, \ldots)$.

Three finite difference schemes are discussed. Scheme a) employs centred approximations to first order spatial derivatives, whereas b) and c) employ non-centred. With schemes a) and b) in passing from time $n\Delta t$ to $(n+1)\Delta t$, the entire grid is worked over before any of the field values are reset. Scheme c) is a sequential method; the value of a field variable is reset at every point as it is calculated, so that the values at point j used in calculating the new values at j+1 are at time n+1 not at time n as in a) and b).

Time derivative approximations are non-centred, forward

$$u_t = \frac{u^{(n+1)} - u^{(n)}}{\Delta t}$$

First order spatial derivative approximations are either centred

$$u_z = \frac{u_{j+1} - u_{j-1}}{2\,\Delta z}$$

or, non-centred — forward $\qquad u_z = \dfrac{u_{j+1} - u_j}{\Delta z}$

or backward $\qquad u_z = \dfrac{u_j - u_{j-1}}{\Delta z}$

Second order spatial derivative approximations are centred

$$u_{zz} = \frac{1}{(\Delta z)^2}\left(u_{j+1} - 2u_j + u_{j-1}\right)$$

Equation (17) is approximated by

$$G_j = \left(\tfrac{1}{2}u_0 + u_1 + \cdots + u_{j-1} + \tfrac{1}{2}u_j\right)\Delta z - 2\frac{c\,w_1}{(\Delta z)^2}$$

$$- \frac{2}{\Delta z}\left(w_1 \Delta_1^2 w + w_2 \Delta_2^2 w + \cdots + w_{j-1}\Delta_{j-1}^2 w + \tfrac{1}{2} w_j \Delta_j^2 w\right)$$

where $\qquad \Delta_j^2 w = w_{j+1} - 2w_j + w_{j-1}$

The surface stress boundary condition applied to equation (15) is approximated by

$$\frac{u_J - u_{J-1}}{\Delta z} = -\frac{1}{c}$$

For each finite difference scheme, there is a computational stability criterion $f(\Delta z, \Delta t, c) \leq k$ which determines the largest allowable value of the time step Δt for any particular values of Δz and c. In discussing the stability of different schemes, the system is reduced to that of equation (15) with a constant w coefficient representing some sort of average value of w, and $u - \frac{z}{w}$ is replaced by u, so that the equation becomes

$$u_t = -w u_z + c u_{zz} \qquad (18)$$

<u>Comparison of finite difference schemes a) and b)</u>

The finite difference approximations to equation (18) are

$$u_j^{(n+1)} = u_j^{(n)} - \frac{w \Delta t}{2 \Delta z}\left(u_{j+1} - u_j\right)^{(n)} + \frac{c \Delta t}{(\Delta z)^2}\left(u_{j+1} - 2u_j + u_{j-1}\right)^{(n)} \qquad a)$$

$$u_j^{(n+1)} = u_j^{(n)} - \left\{\begin{matrix} \frac{w \Delta t}{\Delta z}\left(u_j - u_{j-1}\right)^{(n)} \\ \frac{w \Delta t}{\Delta z}\left(u_{j+1} - u_j\right)^{(n)} \end{matrix}\right\} + \frac{c \Delta t}{(\Delta z)^2}\left(u_{j+1} - 2u_j + u_{j-1}\right)^{(n)} \qquad b)$$

In b) the non-centred approximations to first order spatial derivatives are calculated in the upstream direction, the upper alternative corresponding to w positive and the lower to w negative.

According to Richtmyer, the computational stability criterion for a) is $0 \leq \frac{c \Delta t}{(\Delta z)^2} \leq \frac{1}{2}$. This is independent of the sign and magnitude of w. For b) the criterion is $0 \leq \frac{c \Delta t}{(\Delta z)^2} < \frac{1}{2}$ for sufficiently small Δt. Note that it is impossible to use $\frac{c \Delta t}{(\Delta z)^2} = \frac{1}{2}$ here. The requirement of "sufficiently small Δt "is due to the presence of a term of $O((\Delta t)^{1/2})$. I think it more useful to include this term in the criterion, and require $0 \leq 2\frac{c \Delta t}{(\Delta z)^2} + |w| \frac{\Delta t}{\Delta z} \leq 1$;

$$w \frac{\Delta t}{\Delta z} = w \sqrt{\frac{\alpha \Delta t}{c}}, \quad \alpha = \frac{c \Delta t}{(\Delta z)^2}$$

For all reasonable values of viscosity the first term forms the major contribution, but for the total system of equations (15), (16) and (17), where w is variable, it is observed numerically that as c is decreased, w becomes larger and the second term therefore becomes more important.

It is of interest to note that if the condition on the sign of w is reversed in b) the stability criterion becomes

$$0 \leq 2c\frac{\Delta t}{(\Delta z)^2} - |w|\frac{\Delta t}{\Delta z} \leq 1 ,$$

and for reasonable values of c, the largest allowable Δt is greater than that for scheme a). However, if $\Delta t > \frac{(\Delta z)^2}{2c}$, while also satisfying the above criterion, an apparently converged spatial oscillatory solution is obtained. (A similar phenomenon is described below in connection with scheme a).) If $\Delta t \leq \frac{(\Delta z)^2}{2c}$, a good solution is obtained — good in the sense that it is non-oscillatory and represents steady-state. It has, however, a large truncation error and there are no apparent advantages.

Richtmyer determines the truncation error of schemes a) and b). He obtains for a), $O(\Delta t) + O((\Delta z)^2)$ and for b), $O(\Delta t) + O(\Delta z)$. From this point of view the centred differences should be more accurate than the non-centred. For the whole system of equations it is observed that values of Δt small enough to satisfy the stability criterion do not affect the truncation error. However scheme b) is very sensitive to changes in Δz. For example, for $c = \frac{1}{25}$, a change in Δz from $\frac{1}{20}$ to $\frac{1}{40}$ causes an increase in the maximum value of u of about 5.7%, whereas the corresponding decrease for scheme a) is about 1.7%.

It appears that the u-profiles obtained from b) approach those of a) from below as Δz is decreased.

Using centred difference scheme a) in the total system, keeping Δz constant, altering Δt such that the stability criterion is just satisfied and reducing c, it is observed that below a certain value of c, the equations converge fairly rapidly to a spatially oscillating solution. These oscillations are of small amplitude and of wavelength twice the grid distance, such that each alternate point lies on a smooth curve. They occur in both w and u-fields near the lower boundary. In calculating the magnitudes of the terms in the equation $u_t + w u_z = 1 + c u_{zz}$ from the apparently converged profiles, it is seen that the last three terms are not in balance. This shows that the profiles are not actually converged. As convergence takes several thousand iterations, it is reasonable to print out the fields every few hundred iterations. If in this case, after convergence the fields are printed for a few consecutive iterations, temporal oscillations of period two iterations are seen. This is not a physical phenomenon. Satisfactory convergence is obtained using scheme b) for the same values of c and Δz . Using scheme a) with $\Delta z = \frac{1}{20}$ these oscillations first appeared for $c = \frac{1}{30}$. When Δt was decreased the fields slowly diverged. The program was run until the velocity values were greater than twice the converged values obtained from b), and there was no sign of convergence.

<u>Stability criterion for sequential differencing, scheme c)</u>

The finite difference approximation of scheme c) to equation (18) is

$$u_j^{(n+1)} = u_j^{(n)} - \frac{w\,\Delta t}{\Delta z}\left(u_j^{(n)} - u_{j-1}^{(n+1)}\right) + \frac{c\,\Delta t}{(\Delta z)^2}\left(u_{j+1}^{(n)} - 2\,u_j^{(n)} + u_{j-1}^{(n+1)}\right) \quad \text{c)}$$

where w is a positive constant for upstream differencing.

Let $\quad u_j^{(n)} = Z_n e^{ik\Delta z j}$

$$Z_{n+1} = Z_n - \frac{\Delta t\, w}{\Delta z}\left(Z_n - Z_{n+1} e^{-ik\Delta z}\right)$$

$$+ \frac{c\,\Delta t}{(\Delta z)^2}\left(e^{ik\Delta z} - 2\right) Z_n + \frac{c\,\Delta t}{(\Delta z)^2} e^{-ik\Delta z} Z_{n+1}$$

$$Z_{n+1}\left[1 - (a+b)e^{-ik\Delta z}\right] = Z_n\left[1 - a + b\left(e^{ik\Delta z} - 2\right)\right]$$

where $\quad a = \frac{\Delta t\, w}{\Delta z} \quad , \quad b = \frac{c\,\Delta t}{(\Delta z)^2}$

The amplification matrix G is given by $\quad G Z_n = Z_{n+1}$

Since G is a scalar, the eigenvalue λ of G is equal to G.

$$\lambda = \frac{Z_{n+1}}{Z_n} = \frac{1 - a + b\left(e^{ik\Delta z} - 2\right)}{1 - (a+b)e^{-ik\Delta z}}$$

$$= \frac{\left[1 - a + b\left(e^{ik\Delta z} - 2\right)\right]\left[1 - (a+b)e^{ik\Delta z}\right]}{1 - 2(a+b)\cos k\Delta z + (a+b)^2}$$

$$= \frac{1 - a - 2b + \left(-a + a^2 + 3ab + 2b^2\right)e^{ik\Delta z} - b(a+b)e^{2ik\Delta z}}{1 - 2(a+b)\cos k\Delta z + (a+b)^2}$$

or $$D|\lambda|^2 = (A + B\cos k\Delta z + E\cos 2k\Delta z)^2 + (B\sin k\Delta z + E\sin 2k\Delta z)^2$$

where $A = 1 - a - 2b$, $B = -a + a^2 + 3ab + 2b^2$,

$E = -b(a+b)$, $D = [1 - 2(a+b)\cos k\Delta z + (a+b)^2]^2$

Von Neumann's necessary condition for stability requires that

$|\lambda| \leq 1 + O(\Delta t)$ for all values of k.

When $b = 1$, $A = -(1+a)$, $B = 2a + a^2 + 2 = 1 + A^2$,

$$E = A, \quad D = [1 + 2A\cos k\Delta z + A^2]^2$$

$$D|\lambda|^2 = \left[A + (1+A^2)\cos k\Delta z + A\cos 2k\Delta z\right]^2 + \left[(1+A^2)\sin k\Delta z + A\sin 2k\Delta z\right]^2$$

$$= 2A^2 + (1+A^2)^2 + 2A(1+A^2)\cos k\Delta z + 2A^2\cos 2k\Delta z$$

$$+ 2A(1+A^2)\cos k\Delta z\cos 2k\Delta z + 2A(1+A^2)\sin k\Delta z\sin 2k\Delta z$$

$$= 2A^2 + (1+A^2)^2 + 4A(1+A^2)\cos k\Delta z + 2A^2(2\cos^2 k\Delta z - 1)$$

$$= (1+A^2)^2 + 4A(1+A^2)\cos k\Delta z + 4A^2\cos^2 k\Delta z$$

$$= \left[(1+A^2) + 2A\cos k\Delta z\right]^2$$

$$= D$$

$\therefore |\lambda|^2 = 1$ when $b = 1$

In fact if $|\lambda|^2$ is calculated for various values of the parameters a, b and k using the computer, it is seen that $|\lambda|^2 \leq 1$ for $0 \leq b \leq 1$, for all a and k. Thus the von Neumann necessary condition for stability is satisfied if $0 \leq \frac{c\,\Delta t}{(\Delta z)^2} \leq 1$

A test for sufficiency should also be applied. According to Richtmyer, if G($\Delta t, k$) is a normal matrix, the von Neumann condition is sufficient as well as necessary for stability. Since any scalar is a normal matrix this test is satisfied in the present case. Thus the criterion $0 \leq \frac{c\,\Delta t}{(\Delta z)^2} \leq 1$ is a necessary and sufficient condition for stability. Note that this condition is independent of the vertical velocity w. For the total system a decrease in c was found to give an increase in w. Therefore, assuming the stability criterion to be governed by linear affects alone, this method should be good for small values of c.

If the problem is reduced to that of the u-equation with w any positive constant $u_t + w u_z = 1 + c u_{zz}$, $0 \leq z \leq 1$, together with the boundary conditions $u(0) = 0$, $u_z(1) = -\frac{1}{c}$, using the above finite difference scheme c), convergence should be obtained for $\frac{c\,\Delta t}{(\Delta z)^2} = 1$. This was not found to be the case. For convergence, the value of $\frac{c\,\Delta t}{(\Delta z)^2}$ had to be decreased to some value $K(w)$, an apparently monotonic decreasing function of w.

There are two factors which have not been considered:- initial conditions and boundary conditions. When the system was executed on the computer, using $\frac{c\,\Delta t}{(\Delta z)^2} = 1$, instability was first seen to occur at or near the upper boundary, where the stress condition was applied. The boundary condition was altered to that of a rigid boundary $u(1) = 0$, but the same phenomenon was observed.

Scheme c) is not completely applicable to the total system, since the method relies on the fact that the grid is passed over in the same sense as the vertical velocity w. In calculating velocities at a point j and time $t = n\,\Delta t$, the values of the upstream velocities used are those at time n. For values of c smaller than about $\frac{1}{4}$ there are two vertical cells — upward motion in the upper regions and downward in the lower regions. Thus in working over the grid from bottom to top, in the lower region the upstream velocities are unknown at time n and the values at time n-1 have to be used. This involves a different stability criterion. In this case the finite difference approximation of scheme c) to equation (18) is

$$u_j^{(n+1)} = u_j^{(n)} - \frac{w\,\Delta t}{\Delta z}\left(u_{j+1}^{(n)} - u_j^{(n)}\right) + \frac{c\,\Delta t}{(\Delta z)^2}\left(u_{j+1}^{(n)} - 2\,u_j^{(n)} + u_{j-1}^{(n+1)}\right)$$

where w is a negative constant, for upstream differencing. As before let

$$u_j^{(n)} = Z_n\, e^{ik\Delta z j}$$

$$Z_{n+1} = Z_n - \frac{\Delta t\, w}{\Delta z}\left(e^{ik\Delta z} - 1\right) Z_n + \frac{c\,\Delta t}{(\Delta z)^2}\left(e^{ik\Delta z} - 2\right) Z_n + \frac{c\,\Delta t}{(\Delta z)^2}\, e^{-ik\Delta z}\, Z_{n+1}$$

$$Z_{n+1}\left[1 - b\,e^{-ik\Delta z}\right] = Z_n\left[1 - a\left(e^{ik\Delta z} - 1\right) + b\left(e^{ik\Delta z} - 2\right)\right]$$

where $a = \frac{\Delta t\, w}{\Delta z}$, $b = \frac{c\,\Delta t}{(\Delta z)^2}$

This leads to the amplification matrix

$$G = \lambda = \frac{1 + a - 2b + (-a - ab + 2b^2)e^{ik\Delta z} - b(-a+b)e^{2ik\Delta z}}{1 - 2b\cos k\Delta z + b^2}$$

Using the computer to calculate $|\lambda|^2$ for a range of values of a, b and k, it is easy to determine for what values of the parameters a and b the von Neumann condition is obeyed. There is quite a strong w-dependence as can be seen from the following table.

$-a = -w\frac{\Delta t}{\Delta z}$	$b_{max} = \frac{c\,\Delta t}{(\Delta z)^2}\ max$
0.1	1.0
0.2	0.8
0.3	0.7
0.4	0.6
0.5	0.5
0.6	0.4
0.7	0.3
0.8	0.2
0.9	0.1
1.0	0.0

Given a, b_{max} is the maximum allowable value of b for convergence.

Consistency of scheme c)

The continuous equation and the finite difference approximation to it are respectively

$$u_t \quad + \quad w\, u_z \quad = \quad c\, u_{zz}$$

$$\frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t} + w\frac{\left(u_j^{(n)} - u_{j-1}^{(n+1)}\right)}{\Delta z} = \frac{c\left(u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n+1)}\right)}{(\Delta z)^2} \qquad (19)$$

Expanding the terms $u_j^{(n+1)}$, $u_{j-1}^{(n+1)}$, $u_{j+1}^{(n)}$ in a Taylor series about $\,^{(n)}_{j}$,

$$u_j^{(n+1)} = \left(u + \Delta t\, u_t + \frac{(\Delta t)^2}{2!} u_{tt} + \cdots\right)_j^{(n)}$$

$$u_{j-1}^{(n+1)} = \left(u - \Delta z\, u_z + \frac{(\Delta z)^2}{2!} u_{zz} - \cdots\right)_j^{(n+1)}$$

$$u_{j+1}^{(n)} = \left(u + \Delta z\, u_z + \frac{(\Delta z)^2}{2!} u_{zz} + \cdots\right)_j^{(n)}$$

and substituting these expressions into equation (19) ,we have

$$u_t\left(1 - \frac{c\,\Delta t}{(\Delta z)^2}\right) + w\, u_z = c\, u_{zz} + O\left(\Delta t, \frac{\Delta t}{\Delta z}, \Delta z\right)$$

Since the stability criterion requires that $\frac{c\,\Delta t}{(\Delta z)^2}$ be kept fixed as Δt tends to zero, $\frac{\Delta t}{\Delta z}$ must approach zero with Δt , Consistency requires that the finite difference approximation tends to the continuous equation as Δt tends to zero. It can be seen that the finite difference approximation would be consistent with the equation

$$u_t (1 - k) + w u_z = c u_{zz}$$

where $k = \frac{c \Delta t}{(\Delta z)^2}$

However as we are only interested in a steady-state solution, this inconsistency should not affect the solution.

Comparison of finite difference schemes a), b) and c)

The truncation errors of b) and c) are the same; if both their stability criteria are satisfied, they produce identical results. That of a) is considerably less, being $O((\Delta z)^2)$ as opposed to $O(\Delta z)$. The times of convergence for a) and b) are comparable and rather large, whereas for c) the convergence time is reduced by a factor of about 20 for values of viscosity which are not too small, say for $c \geq \frac{1}{30}$. For small values of c, scheme c) has one unfortunate characteristic. If the value of Δt chosen is slightly larger than the maximum allowable for convergence, the program will run for many iterations, about 1000, before the velocity fields show any indication that ultimate convergence will not be obtained; there is then a sudden ~~a sudden~~ build up of the velocities and machine overflow occurs. With scheme b) if the stability criterion is violated, overflow occurs almost immediately and very little time is lost.

IV. THE BALANCE OF FORCES OBTAINED IN THE LIMITING CASE OF SMALL VISCOSITY

For small values of the eddy viscosity parameter c it is difficult to obtain convergence for two reasons. The calculated computational stability criterion expressing the maximum allowable value of time step in terms of c and the grid distance Δz , say $0 \leq \frac{c\,\Delta t}{(\Delta z)^2} \leq 1$, is apparently incomplete. According to the above stability theory, if c were decreased, Δt could be increased. Unfortunately this was not always found to be the case; there seems also to be some dependence of Δt on the magnitude of the vertical velocity. This extra condition on Δt is thought to be due somehow to the application of the upper boundary condition. As the eddy viscosity is decreased the vertical velocity is observed to increase and Δt has therefore to be kept at a fairly small value for stability. The other reason is on physical grounds. Since the viscous forces are small, the momentum exchange rate is also small; it takes a long time for the dissipation of any disturbance in the fluid. In this case the disturbance occurs in the form of a wind stress at the surface. Since it takes a long time for convergence, the only such case considered was $c = \frac{1}{40}$, $\nu = 10.7$. This case was run for about 160,000 iterations, about 100mins. of computer time, and only near convergence was obtained. The increments in velocities over a substantial time step were small and decreasing but so slowly that it was thought not to be worth while continuing the calculation. The steady-state converged values for the velocity fields could be obtained by extrapolation using semi-log graph paper. There

FIG: 1(a) West-east velocity as a function of depth for a small value of the viscosity parameter $c = \frac{\nu}{H^2 (\rho \beta)^{1/3}} = \frac{1}{40}$.

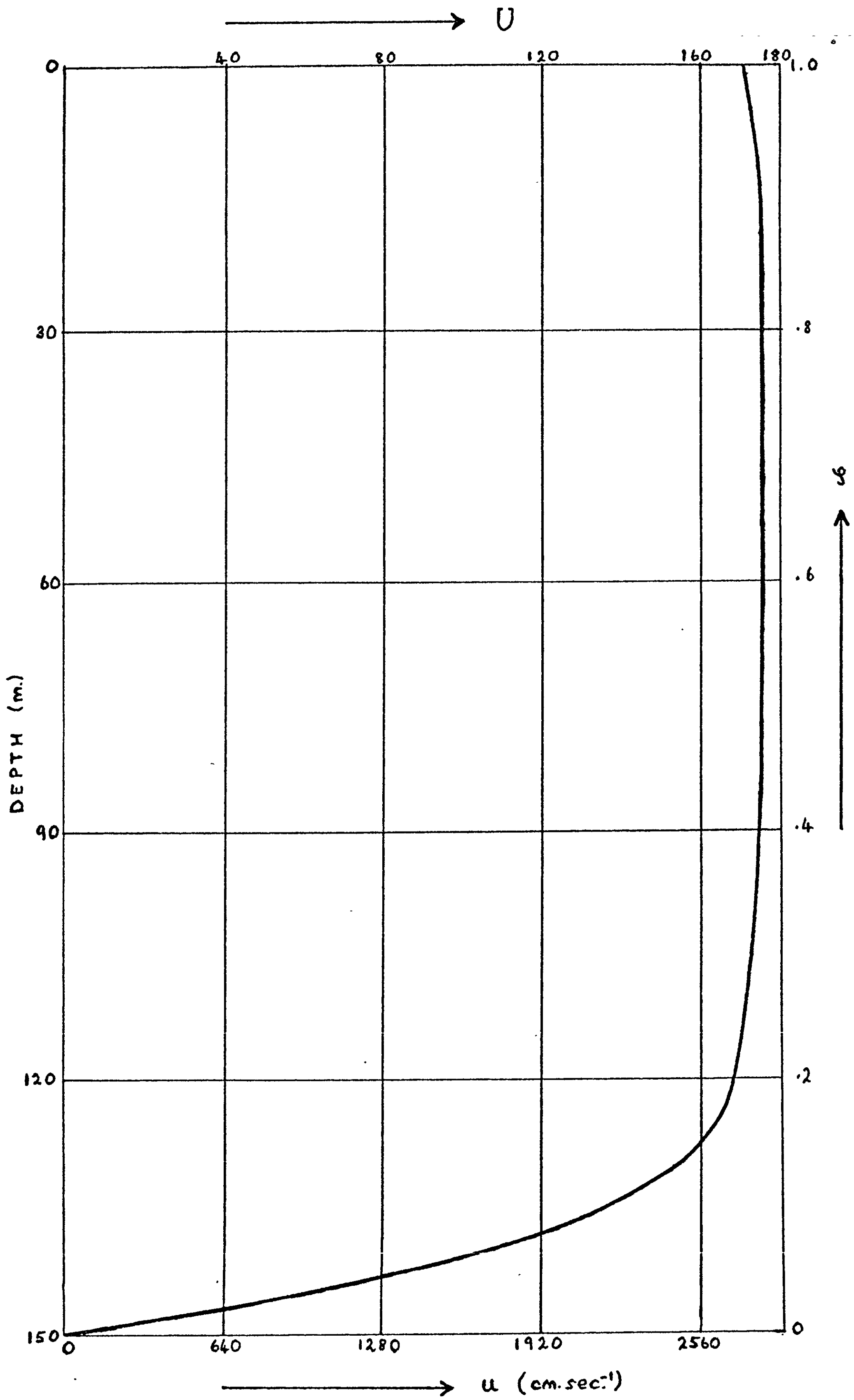


FIG: 1(b) Vertical velocity as a function of depth for a small value of the viscosity parameter $c = \frac{\nu}{H^2 (P\beta)^{1/3}} = \frac{1}{40}$

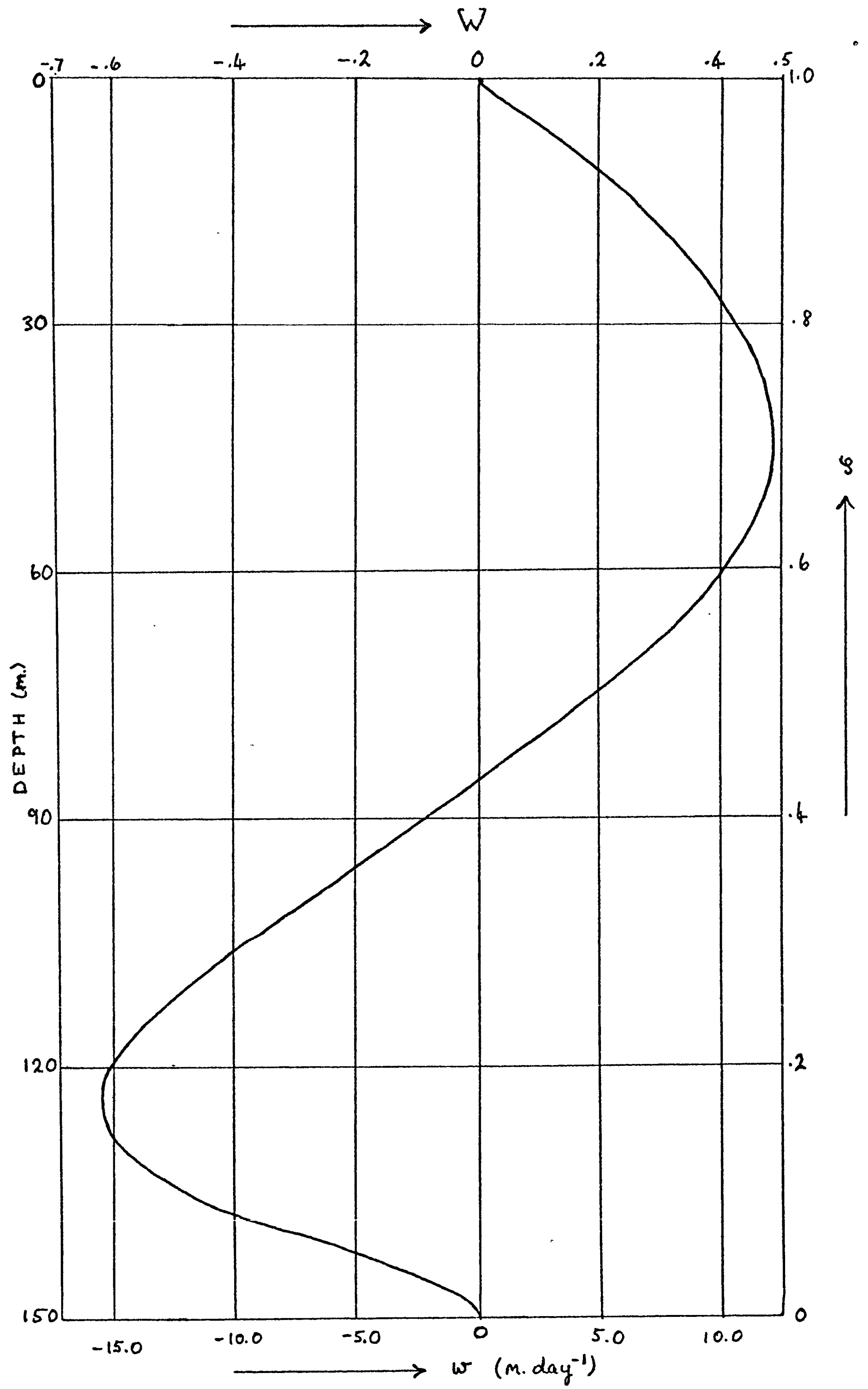


TABLE I

MAGNITUDES OF TERMS IN THE EQUATION

$$U_T + WU_\xi = 1 + cU_{\xi\xi}$$

ξ	WU_ξ	$cU_{\xi\xi}$
0.025	- 133.92	
0.05		
0.075	- 343.74	- 344.75
0.1		
0.125		
0.15	- 147.56	
0.175		
0.2	- 55,02	- 55.96
0.225		
0.25	- 19,17	
0.275		
0.3	- 6,64	
0.325	- 3,85	
0.35	- 2,17	
0.375	- 1,12	- 2.16
0.4	- 0.47	- 1.44
0.425	- 0,05	- 1.04
0.45	0.27	- 0.72
0,475		
0.5		- 0.60
0.525	0.70	- 0.32
0.55	0,75	- 0.24
0.575	0.78	- 0.20
0.6	0.78	- 0.20
0.625	0.77	- 0.20
0.65		- 0.32
0.675	0.59	- 0.36
0.7	0,43	- 0.56
0.725	0.16	- 0.84
0.75	- 0,25	- 1.24
0.775	- 0.81	- 1.76
0.8		
0.825	- 2.43	- 3.44
0.85		
0.875		
0.9	- 4.98	
0.925	- 5,04	- 6.00
0.95	- 4.26	- 5.32
0.975	- 2.53	
1.0	0,00	

TABLE II

MAGNITUDES OF TERMS IN THE EQUATION

$$W_{T\vartheta} + W W_{\vartheta\vartheta} - W_{\vartheta}^{2} - U = -G(1) + c\, W_{\vartheta\vartheta\vartheta}$$

ϑ	$W_{\vartheta}^{2} - W W_{\vartheta\vartheta}$	$c\, W_{\vartheta\vartheta\vartheta}$	U	$G(1) - U$
0.025	25.4		41.79	141.64
0.05	34.2	74.95	80.46	102.97
0.075	40.6	36.17	111.99	71.43
0.1	42.6	8.79	134.91	48.51
0.125	39.2	- 6.3	150.21	33.21
0.15	33.4	- 11.41	159.91	23.52
0.175	25.8	- 11.06	165.88	17.54
0.2	20.6	- 9.00	169.55	13.88
0.225	17.0	- 6.97	171.81	
0.25	14.5	- 5.43	173.24	10.19
0.275	12.8	- 4.32	174.16	
0.3	11.6	- 3.52	174.77	8.65
0.325	10.7	- 2.91	175.19	8.23
0.35	10.0	- 2.44	175.50	
0.375	9.5	- 2.05	175.72	7.70
0.4	9.0	- 1.71	175.89	
0.425	8.6	- 1.57	176.02	
0.45	9.0	- 1.57	176.13	7.29
0.475	8.8	- 1.50	176.22	
0.5	8.6	- 1.39	176.30	7.13
0.525	8.4	- 1.24	176.37	
0.55	8.2	- 1.08	176.43	7.00
0.575	8.0	- 0.90	176.48	
0.6	7.7	- 0.71	176.53	6.89
0.625	7.5	- 0.51	176.57	
0.65	7.2	- 0.30	176.61	6.81
0.675	7.0	- 0.10	176.64	
0.7	6.8	0.10	176.66	6.76
0.725	6.6	0.27	176.67	6.75
0.75	6.4	0.41	176.66	6.76
0.775	6.3	0.52	176.62	
0.8	6.3	0.60	176.53	
0.825	6.3	0.66	176.38	7.05
0.85	6.5	0.75	176.14	
0.875	6.7	0.91	175.79	7.63
0.9	6.9	1.18	175.31	
0.925	7.1	1.61	174.68	8.74
0.95	7.3	2.19	173.90	
0.975	7.5		172.99	10.44
1.0	7.7		171.99	11.44

TABLE III

MAGNITUDES OF TERMS IN THE EQUATION

$$W_{T\xi\xi} + \left(W W_{\xi\xi} - W_\xi^2\right)_\xi - U_\xi = c\, W_{\xi\xi\xi\xi}$$

ξ	$\left(W_\xi^2 - W W_{\xi\xi}\right)_\xi$	$c\, W_{\xi\xi\xi\xi}$	U_ξ
0.025	352.0		1546.8
0.05	256.0	- 1551.2	1261.2
0.075	80.0	- 1095.0	916.8
0.1	- 136.0	- 603.6	612.0
0.125	- 232.0	- 204.3	388.0
0.15	- 304.0	13.9	238.8
0.175	- 208.0	82.5	146.8
0.2	- 145.2	81.0	90.4
0.225	- 98.8	61.8	57.2
0.25	- 68.0	44.2	36.8
0.275	- 48.0	32.2	24.4
0.3	- 34.8	24.2	16.8
0.325	- 29.2	18.9	12.4
0.35	- 20.0	15.8	8.8
0.375	- 19.2	13.4	6.8
0.4	- 16.2	5.48	5.2
0.425	15.3	0.02	4.4
0.45	15.3	0.02	4.4
0.475	- 8.0	2.80	3.6
0.5	- 7.6	4.74	3.2
0.525	- 8.0	5.69	2.8
0.55	- 8.4	6.37	2.4
0.575	- 9.2	7.20	2.0
0.6	- 9.6	7.72	2.0
0.625	- 10.0	8.12	1.6
0.65	- 10.0	8.40	1.6
0.675	- 9.6	8.13	1.2
0.7	- 9.2	7.81	0.8
0.725	- 8.0	6.91	0.4
0.75	- 6.4	5.63	- 0.4
0.775	- 3.6	4.40	- 1.6
0.8	- 1.2	2.96	- 3.6
0.825	2.4	2.59	- 6.0
0.85	5.2	3.49	- 9.6
0.875	7.2	6.24	- 14.0
0.9	8.4	11.1	- 19.2
0.925	8.4	17.1	- 25.2
0.95	8.0	23.3	- 31.2
0.975	7.6		- 36.4
1.0	7.2		- 40.0

was no significant change in the relative order of magnitude of the individual terms in the equations over the last few ten thousand iterations, and the primary concern in this calculation was to determine which terms contributed to the balance. This then was achieved.

The u and w-profiles at the last print-out are plotted in figures 1(a) and 1(b), and it is seen that the u-profile exhibits interesting boundary-layer characteristics. In the calculation of the magnitudes of the different terms the equations used are

$$U_T + W U_\zeta = 1 + c\, U_{\zeta\zeta}$$

$$W_{T\zeta} + W W_{\zeta\zeta} - W_\zeta^2 - U = -G(1) + c\, W_{\zeta\zeta\zeta}$$

The magnitudes are tabulated in tables I and II. In the second equation here, a striking balance occurs between U and G over the greater part of the fluid. In scaling considerations there is no way of determining the constant G(1), so it is more useful to differentiate this equation with respect to ζ and obtain

$$W_{T\zeta\zeta} + \left(W W_{\zeta\zeta} - W_\zeta^2\right)_\zeta - U_\zeta = c\, W_{\zeta\zeta\zeta\zeta}$$

The magnitudes of the terms in this equation appear in table III.

Scaling of the steady-state equations

Let us revert to the dimensional form of the steady-state equations.

$$w u_z = P + \nu u_{zz} \qquad (20)$$

$$w_z w_{zz} - w w_{zzz} + \beta u_z = - \nu w_{zzzz} \qquad (21)$$

The total depth may be divided into three regions, the lower boundary layer, the interior and the upper boundary layer. The balance of forces is discussed in each of the three layers.

Upper boundary layer

The surface stress boundary condition is

$$(u_z)_{z=H} = - \frac{\tau}{\rho \nu} = - \frac{PH}{\nu}$$

This is the order of magnitude of u_z throughout the upper boundary layer. Also if z_u is the height scale then $u_z \sim \frac{\Delta u}{z_u}$ where Δu is of order $u_{max} - u_{surface}$. Then Δu may be defined by $\Delta u = \frac{HP}{\nu} z_u$. (22)
If in equation (21) w is scaled by w_u and w, u and z are replaced by the non-dimensional w, u and z of order unity, multiplied by the scale factors w_u, Δu and z_u, the equation becomes

$$\frac{w_u^2}{z_u^3} \left(w_z w_{zz} - w w_{zzz} \right) + \beta \frac{HP}{\nu} u_z = - \frac{\nu w_u}{z_u^4} w_{zzzz}$$

The numerical calculation shows all terms to be equally important here. Therefore the scale factors w_u and z_u may be defined by the relations

$$\frac{w_u^2}{z_u^3} = \beta \frac{HP}{\nu} = \frac{\nu w_u}{z_u^4} \qquad (23)$$

If equation (20) is likewise non-dimensionalized it becomes

$$w_u \frac{HP}{\nu} w u_z = P + \frac{\nu \Delta u}{z_u^2} u_{zz}$$

Here the first and last terms balance and the pressure gradient force term is smaller, suggesting the relation

$$w_u \frac{HP}{\nu} = \nu \frac{\Delta u}{z_u^2}$$

This relation is consistent with (22) and (23). From (23)

$$w_u = \frac{\nu}{z_u}$$

$$w_u = \beta \frac{HP}{\nu^2} z_u^4$$

$$\Longrightarrow \quad z_u^5 = \frac{\nu^3}{\beta HP}$$

$$w_u^5 = \nu^2 \beta HP$$

and from (22) $\quad (\Delta u)^5 = \frac{(HP)^4}{\nu^2 \beta}$

Thus the upper boundary layer is fully determined within itself. Adopting the choice of H and P from Charney (1960), $H = 1.5\times10^4$ cm., $P = 3\times10^{-5}$ dyne cm.$^{-2}$, $\beta = 2.28\times10^{-13}$ cm.$^{-1}$sec.$^{-1}$ and $c = \frac{1}{40} \Rightarrow \nu = 10.7$ cm.^{2}sec.$^{-1}$, the values for z_u, w_u and Δu are, $z_u = 1.6\times10^3$ cm., $w_u = 6.5\times10^{-3}$ cm.sec.$^{-1}$ and $\Delta u = 69$ cm.sec.$^{-1}$ These values compare well with the numerical calculation.

The ratio of the pressure gradient to the inertial term in equation (20) is $\frac{P\nu}{w_u HP} \sim 0.11$. The neglect of the pressure gradient term is therefore valid here.

Lower boundary layer

In this region w, u and z are scaled by w_b, u_b and z_b respectively.

Replacing w, u and z by the non-dimensional w, u and z of order unity, multiplied by the appropriate scale factors, equations (20) and (21) become

$$\frac{w_b u_b}{z_b} \; w u_z = P + \frac{\nu u_b}{z_b^2} \; u_{zz}$$

$$\frac{w_b^2}{z_b^3}\left(w_z w_{zz} - w\, w_{zzz}\right) + \frac{\beta u_b}{z_b} \; u_z = - \frac{\nu w_b}{z_b^4} \; w_{zzzz}$$

Again all terms in equation (21) contribute to the balance, suggesting the following relations between the scaling factors.

$$\frac{w_b^2}{z_b^3} = \frac{\beta u_b}{z_b} = \frac{\nu w_b}{z_b^4} \qquad (24)$$

In (20) the inertial and viscous terms are comparable and the pressure gradient term is very much smaller. Therefore we require

$$\frac{w_b u_b}{z_b} = \frac{\nu u_b}{z_b^2}$$

This relation is already included in (24). There are therefore three unknowns and only two relations between them. From (24) we have

$$w_b = \frac{\nu}{z_b}$$

$$u_b = \frac{\nu^2}{\beta z_b^4}$$

Interior

In the interior the vertical scale height is the total depth H, w is scaled by W and u by ΔU. As before replacing w, u and z by the non-dimensional w, u and z of order unity, multiplied by the scale factors $W, \Delta U, H$, equation (21) becomes

$$\frac{W^2}{H^3}\left(w_z w_{zz} - w w_{zzz}\right) + \frac{\beta \Delta U}{H} u_z = -\frac{\nu W}{H^4} w_{zzzz}$$

$$\frac{W^2}{H^2 \beta \Delta U}\left(w_z w_{zz} - w w_{zzz}\right) + u_z = -\frac{\nu W}{H^3 \beta \Delta U} w_{zzzz} \qquad (25)$$

From the numerical calculation $W \sim 2.7\times10^{-2}$ cm.sec.$^{-1}$, $\Delta U \sim 80$ cm.sec.$^{-1}$

$$\varepsilon = \frac{W^2}{H^2 \beta \Delta U} \sim 0.18$$

$$\delta = \frac{\nu W}{H^3 \beta \Delta U} \sim 4.6 \times 10^{-3}$$

$$b = \frac{\delta}{\varepsilon^3} \sim 0.84$$

$$\delta = O(\varepsilon^3)$$

The variables may be expanded in a power series of ε,

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots$$

Substituting in equation (25) and equating coefficients of ε^0,

$$u_{0_z} = 0 \qquad (26)$$

$$u_0 = \text{constant}$$

This appears in the profile of u in figure 1(a). Equating coefficients of ε^1

$$w_{0_z} w_{0_{zz}} - w_0 w_{0_{zzz}} + u_{1_z} = 0 \qquad (27)$$

In non-dimensional form equation (20) is

$$\frac{W \Delta U}{H} w u_z = P + \frac{\nu \Delta U}{H^2} u_{zz}$$

$$w u_z = \frac{PH}{W \Delta U} + \frac{\nu}{WH} u_{zz} \qquad (28)$$

$$\gamma = \frac{PH}{W \Delta U} \sim 0.21$$

$$\gamma = O(\varepsilon)$$

$$\lambda = \frac{\nu}{WH} \sim 2.6 \times 10^{-2}$$

$$\lambda = O(\varepsilon^2)$$

$$a = \frac{\gamma}{\varepsilon} = \frac{PH^3 \beta}{W^3} \sim 1.2$$

$$b = \frac{\lambda}{\varepsilon^2} = \frac{\nu H^3 \beta^2 (\Delta U)^2}{W^5} \sim 0.84$$

As before expanding the variables in powers of ε , substituting in equation (28) and equating coefficients of ε^0,

$$w_0 u_{0_z} = 0$$

This is in agreement with (26) . Equating coefficients of ε^1,

$$w_1 u_{0_z} + w_0 u_{1_z} = a \qquad (29)$$

From (27) and (29)

$$w_0 w_{0_z} w_{0_{zz}} - w_0^2 w_{0_{zzz}} + a = 0 \qquad (30)$$

This equation (30) determines the zeroth order w-field in the interior, from which by equation (29) the first order interior u-field could be calculated. However I am unable to solve equation (30) except for the case of zero a, which is of little interest here as for finite ε it only occurs when the pressure gradient force or the depth is zero.

If a and b are set equal to unity, the scale factors for the interior are given by

$$\Delta U = \frac{H (P\beta)^{\frac{5}{6}}}{\beta\sqrt{\nu}} \sim 100 \text{ cm.sec}^{-1}$$

$$W = H (P\beta)^{\frac{1}{3}} \sim 2.9 \times 10^{-2} \text{cm.sec}^{-1}$$

Reverting to the two boundary layers, it is seen that the relative importance of the west-east pressure gradient term in the two layers has not been considered. In both layers it is small compared with other terms in the equation; but if in the upper boundary layer the ratio of the pressure gradient term to the inertial term is $\alpha = \frac{P\nu}{w_u H P} = \frac{\nu}{w_u H} < 1$

then the same ratio in the lower boundary layer $\frac{P z_b}{\varpi_b u_b}$ can be expressed as α^n where $n > 1$. The numerical calculation of the magnitudes of the terms suggests the choice of $n = 3$. Then we have

$$\frac{P z_b}{\varpi_b u_b} = \left(\frac{\nu}{\varpi_u H}\right)^3 \qquad (31)$$

The other relations between the scaling factors of the lower boundary layer were

$$\varpi_b = \frac{\nu}{z_b}$$

$$u_b = \frac{\nu^2}{\beta z_b^4}$$

Substituting in (31), $z_b^6 = \left(\frac{\nu^2}{\varpi_u H}\right)^3 \frac{1}{P\beta}$

For the case of $\nu = 10.7 \text{ cm}^2\text{sec}^{-1}$ the magnitudes of the scaling factors thus obtained are $z_b = 7.8 \times 10^2$ cm., $\varpi_b = 1.4 \times 10^{-2}$ cm. sec.$^{-1}$, $u_b = 1.3 \times 10^3$ cm. sec.$^{-1}$

V. THE EFFECT OF IMPOSING A PRESSURE GRADIENT FORCE INDEPENDENT OF THE SURFACE WIND STRESS

It is possible for a pressure gradient force to exist independently of the surface wind stress. The effect of this can be investigated by changing the surface stress condition to $U_\varsigma = -\frac{1}{c\alpha}$, while leaving the pressure term unaltered. Unfortunately this involves the use of another free parameter α. Four runs were made using values of c of

$\frac{1}{25}$ and $\frac{1}{30}$ with two values of α , α = 2, .5 for both cases. A u-velocity profile bearing quantitative similarity to the observed profiles was obtained for the case $c = \frac{1}{30}$, α = .5. The u-profiles are shown in figures 2(a) and 2(b), and the w-profiles corresponding to the more realistic profiles of u, those in figure 2(a), appear in figure 3.

FIG: 2(a) West-east velocity as a function of depth for different values of the surface stress parameter α and the viscosity parameter $c = \frac{\nu}{H^2 (P\beta)^{1/3}}$

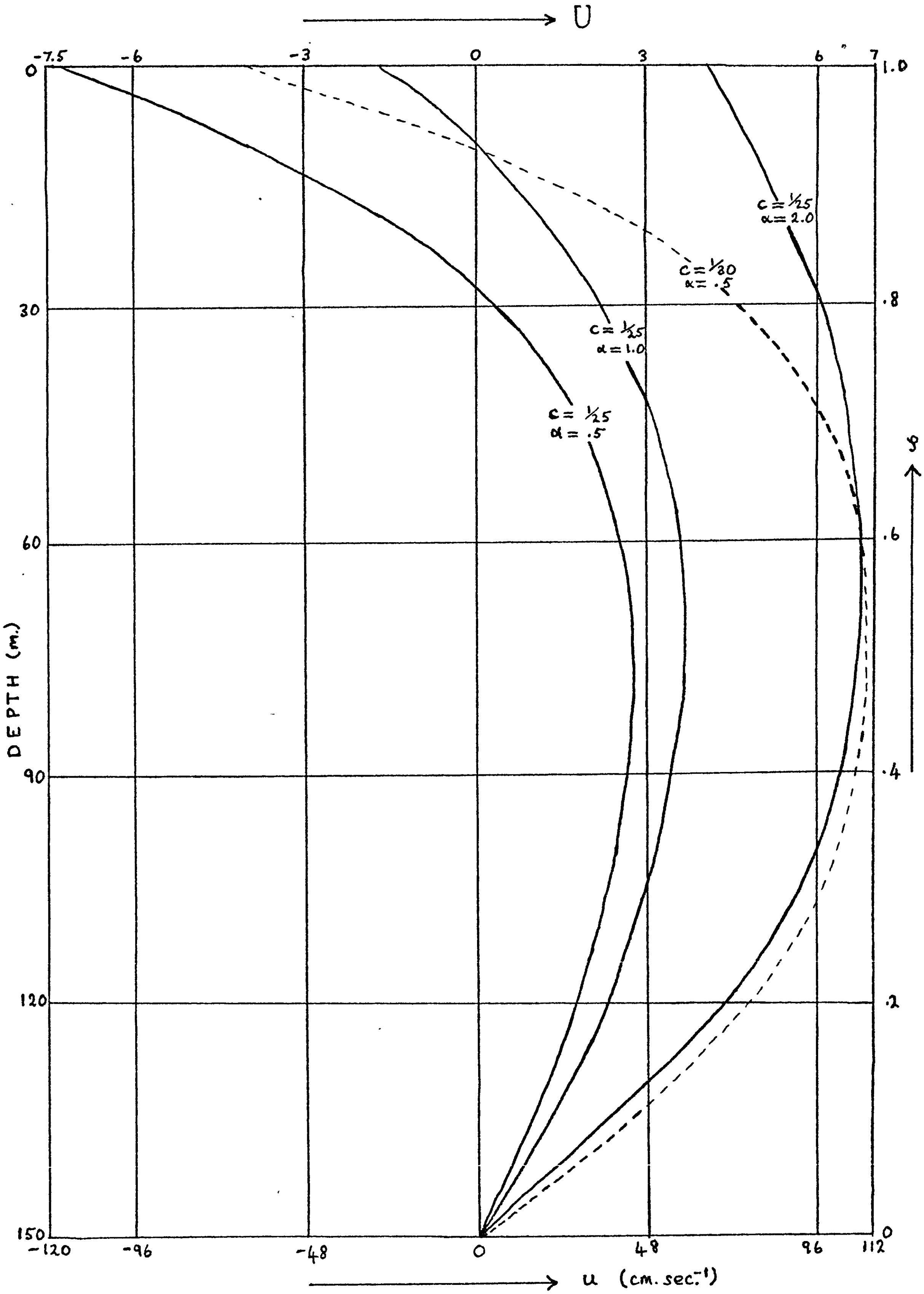


FIG: 2(b) West-east velocity as a function of depth for different values of the surface stress parameter α and for viscosity parameter $c = \frac{\nu}{H^2 (\beta\beta)^{1/3}} = \frac{1}{30}$

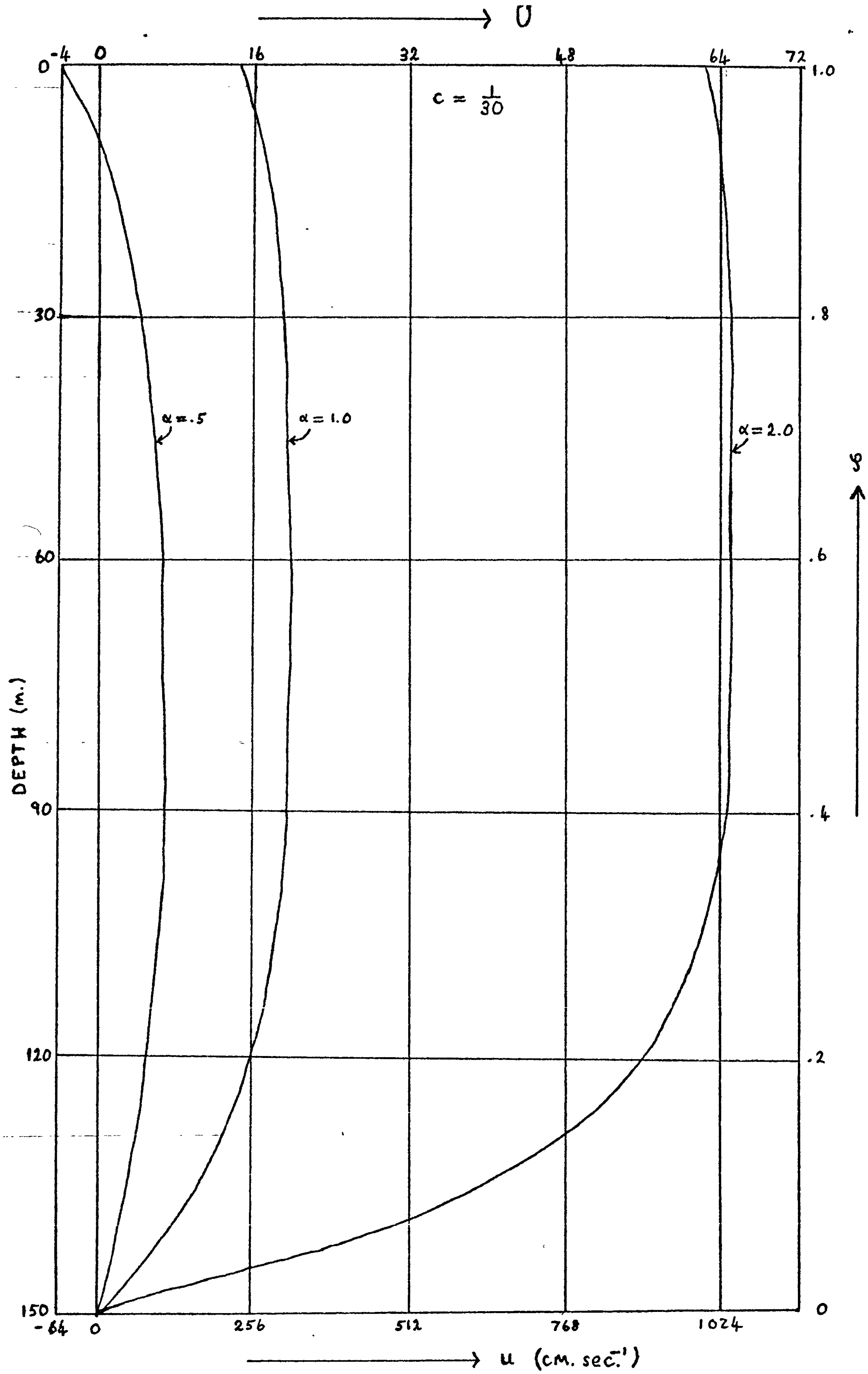
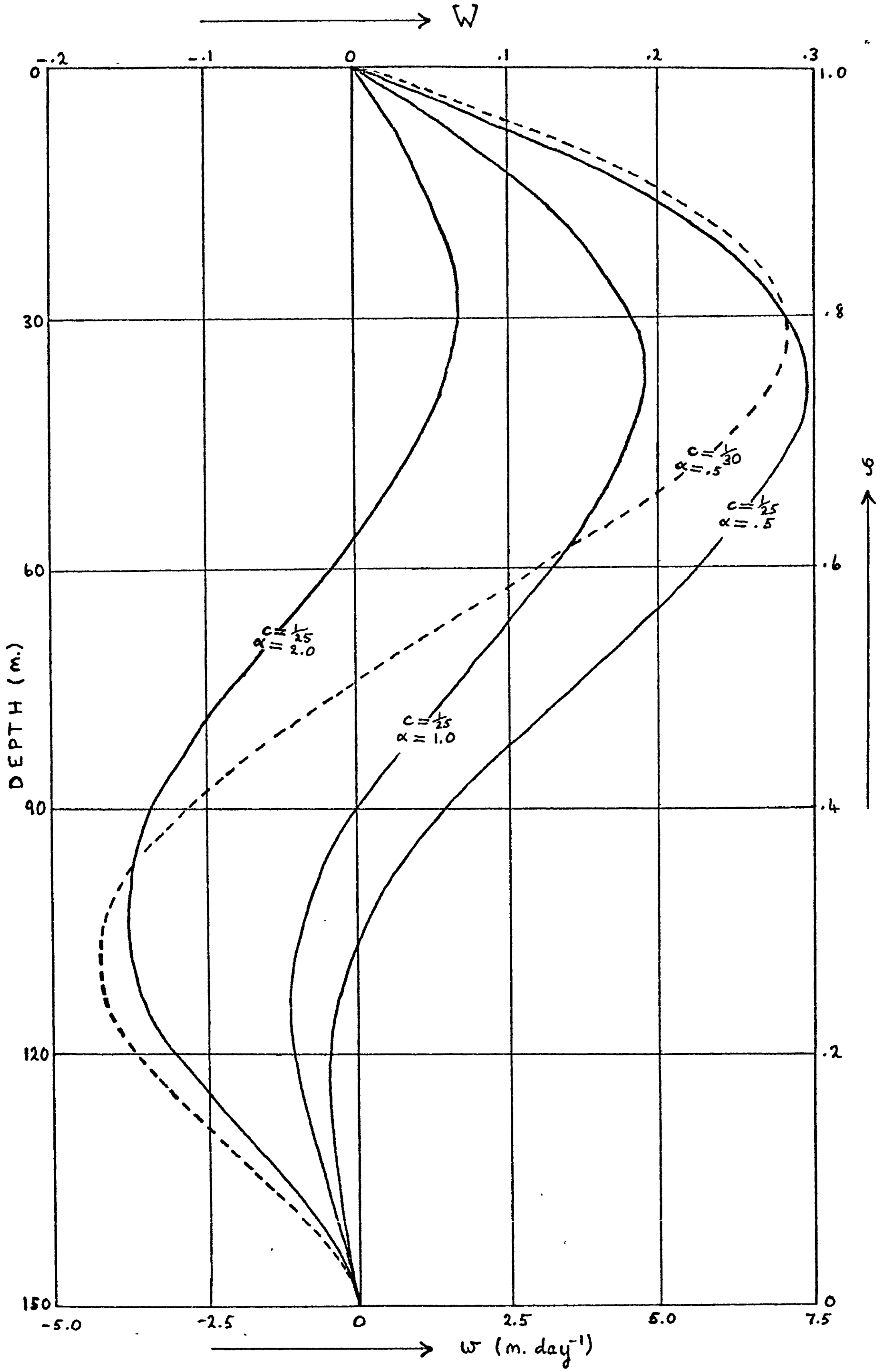


FIG: 3 Vertical velocity as a function of depth for different values of the surface stress parameter α and the viscosity parameter $c = \frac{\nu}{H^2 (P\beta)^{1/3}}$



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